CHAPTER 8

Complex Number-I

IOTA

$$i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$$

So, $i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i,\ i^{4n+4} = 1$

In other words, $i^n = \begin{cases} (-1)^{n/2}, \text{ if } n \text{ is an even integer} \\ (-1)^{\frac{n-1}{2}} \cdot i, \text{ if } n \text{ is an odd integer} \end{cases}$.

The Complex Number System

$z = a + ib$, then $a - ib$ is called conjugate of z and is denoted by $\bar{z}$

Equality in Complex Number

$z_1 = z_2 \Rightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $Im\,(z_1) = Im\,(z_2)$.

Conjugate Complex

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$ i.e. $\bar{z} = a - ib$.

Note:

(*i*) $z + \bar{z} = 2\,\text{Re}(z)$

(*ii*) $z - \bar{z} = 2i\,\text{Im}(z)$

(*iii*) $z\bar{z} = a^2 + b^2$ which is real

(*iv*) If z is purely real then $z - \bar{z} = 0$

(*v*) If z is purely imaginary then $z + \bar{z} = 0$

Important Properties of Conjugate

(*a*) $\overline{(\bar{z})} = z$

(*b*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*c*) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(*d*) $\overline{z_1\, z_2} = \bar{z}_1 \cdot \bar{z}_2$

(*e*) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2};\ z_2 \neq 0$

(*f*) If $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

Important Properties of Modulus

(*a*) $|z| \geq 0$

(*b*) $|z| \geq \text{Re}(z)$

(*c*) $|z| \geq \text{Im}\,(z)$

(*d*) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(*e*) $z\bar{z} = |z|^2$

(*f*) $|z_1 z_2| = |z_1| \cdot |z_2|$

(*g*) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|},\ z_2 \neq 0$

(*h*) $|z^n| = |z|^n$

(*i*) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1 \bar{z}_2)$
or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

(*j*) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$

(*k*) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
[Triangle Inequality]

(*l*) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$
[Triangle Inequality]

(*m*) If $\left|z + \dfrac{1}{z}\right| = a\ (a > 0)$, then $\max |z| = \dfrac{a + \sqrt{a^2 + 4}}{2}$

and $\min |z| = \dfrac{1}{2}\left(\sqrt{a^2 + 4} - a\right)$.

Important Properties of Amplitude

(*a*) $\text{amp}\,(z_1 \cdot z_2) = \text{amp}\,z_1 + \text{amp}\,z_2 + 2\,k\pi\ ;\ k \in I$.

(*b*) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}\,z_1 - \text{amp}\,z_2 + 2\,k\pi\ ;\ k \in I$.

(*c*) $\text{amp}\,(z^n) = n\,\text{amp}(z) + 2k\pi$, where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(*d*) $\log(z) = \log(re^{i\theta}) = \log r + i\theta = \log |z| + i\,\text{amp}\,(z)$.

Demoivre's Theorem

Case I: If n is any integer then

(*i*) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(*ii*) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)(\cos\theta_4 + i\sin\theta_4) \ldots (\cos\theta_n + i\sin\theta_n) = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots \theta_n)$

Case II: If $p, q \in Z$ and $q \neq 0$ then $(\cos\theta + i\sin\theta)^{p/q}$

$$= \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where $k = 0, 1, 2, 3 \ldots q - 1$.

Cube Root of Unity

(*i*) The cube roots of unity are 1, $\dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}$.

(*ii*) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^t + \omega^{2t} = 0$; where $t \in I$ but is not the multiple of 3.

(c) $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$a^3 + b^3 = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$a^3 - b^3 = (a - b)(a - \omega b)\,(a - \omega^2 b)$

$x^2 + x + 1 = (x - \omega)(x - \omega^2)$

Square root of Complex Number

$$\sqrt{a + ib} = \pm\left\{\frac{\sqrt{|z| + a}}{2} + i\frac{\sqrt{|z| - a}}{2}\right\} \text{ for } b > 0$$

and $\pm\left\{\frac{\sqrt{|z| + a}}{2} - i\frac{\sqrt{|z| - a}}{2}\right\}$ for $b < 0$ where $|z| = \sqrt{a^2 + b^2}$.